Review, Identities, Theorems, Formulas and Tables for Math 3710 and 4710

Let n, $\overline{n}$, m, $\overline{m}$, k, $\overline{k}$, l, p, q and $\overline{q}$ be nonnegative integers, unless stated otherwise. Let z be a nonnegative real number, unless stated otherwise.

Trigonometric Identities

1. $\sin a\cos b = \frac{1}{2}[\sin(a+b)+\sin(a-b)]$ 2. $\sin a\sin b = \frac{1}{2}[\cos(a-b)-\cos(a+b)]$
3. $\cos a\cos b = \frac{1}{2}[\cos(a+b)+\cos(a-b)]$

Hyperbolic Functions

4. $\sinh x = \frac{e^x-e^{-x}}{2}$ 5. $\cosh x = \frac{e^x+e^{-x}}{2}$ 6. $\tanh x = \frac{\sinh x}{\cosh x}$ 7. $\coth x = \frac{\cosh x}{\sinh x}$ 8. $\operatorname{sech} x = \frac{1}{\cosh x}$
9. $\operatorname{csch} x = \frac{1}{\sinh x}$

10. $\sinh(-x) = -\sinh x$ 11. $\cosh(-x) = \cosh x$ 12. $\cosh^2 x - \sinh^2 x = 1$ 13. $1-\tanh^2 x = \operatorname{sech}^2 x$
14. $\sinh(x\pm y) = \sinh x\cosh y \pm \cosh x\sinh y$ 15. $\cosh(x\pm y) = \cosh x\cosh y \pm \sinh x\sinh y$

16. $\frac{d}{dx}(\sinh x) = \cosh x$ 17. $\frac{d}{dx}(\cosh x) = \sinh x$ 18. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ 19. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
20. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x\tanh x$ 21. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x\coth x$

Integrals

22. $\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$ 23. $\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$

24. $\int x^2\sin ax\,dx = \frac{2}{a^3}\cos ax + \frac{2}{a^2}x\sin ax - \frac{1}{a}x^2\cos ax + C$

25. $\int x^2\cos ax\,dx = -\frac{2}{a^3}\sin ax + \frac{2}{a^2}x\cos ax + \frac{1}{a}x^2\sin ax + C$

26. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$ 27. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

Definite Integrals

28. $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} \frac{L}{2}, & \text{if } n=m\neq 0 \\ 0, & \text{otherwise} \end{cases}$ 29. $\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} L, & \text{if } n=m=0 \\ \frac{L}{2}, & \text{if } n=m\neq 0 \\ 0, & \text{otherwise} \end{cases}$

30. For positive integers n and m, $\int_0^L \cos\frac{(2n-1)\pi x}{2L}\cos\frac{(2m-1)\pi x}{2L}\,dx = \begin{cases} \frac{L}{2}, & \text{if } n=m \\ 0, & \text{otherwise} \end{cases}$

31. For positive integers n and m, $\int_0^L \sin\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi x}{2L}\,dx = \begin{cases} \frac{L}{2}, & \text{if } n=m \\ 0, & \text{otherwise} \end{cases}$

32. $\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} L, & \text{if } n=m\neq 0 \\ 0, & \text{otherwise} \end{cases}$ 33. $\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 2L, & \text{if } n=m=0 \\ L, & \text{if } n=m\neq 0 \\ 0, & \text{otherwise} \end{cases}$

34. $\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$

35. For $0 < \alpha_1 < \alpha_2 < \cdots$ zeros of $J_z(x)$ and $z \geq 0$, $\int_0^a J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\overline{m}} r}{a})\, r\, dr = \begin{cases} \frac{a^2}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \\ 0, & \text{otherwise} \end{cases}$

36. For $0 < \beta_1 < \beta_2 < \cdots$ zeros of $J_0'(x)$, $\int_0^a J_0(\frac{\beta_m r}{a})\, r\, dr = 0$ and

$$\int_0^a J_0(\frac{\beta_m r}{a}) J_0(\frac{\beta_{\overline{m}} r}{a})\, r\, dr = \begin{cases} \frac{a^2}{2} J_0^2(\beta_m), & \text{if } \overline{m} = m \\ 0, & \text{otherwise} \end{cases}$$

37. For $0 < \beta_1 < \beta_2 < \cdots$ zeros of $J_z'(x)$ and $z > 0$,

$$\int_0^a J_z(\frac{\beta_m r}{a}) J_z(\frac{\beta_{\overline{m}} r}{a})\, r\, dr = \begin{cases} \frac{a^2}{2}[J_z^2(\beta_m) - J_{z-1}(\beta_m) J_{z+1}(\beta_m)], & \text{if } \overline{m} = m \\ 0, & \text{otherwise} \end{cases}$$

38*. $$\int_0^\pi P_n^m(\cos\phi) P_{\overline{n}}^m(\cos\phi) \sin\phi\, d\phi = \begin{cases} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \\ 0, & \text{otherwise} \end{cases}$$

39*. $$\int_{-1}^1 P_n^m(s) P_{\overline{n}}^m(s)\, ds = \begin{cases} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \\ 0, & \text{otherwise} \end{cases}$$

* For $m = 0$, $P_k^m(s) = P_k(s)$ and $\frac{(n+m)!}{(n-m)!} = 1$.

Definite Double Integrals

40. $$\int_0^L \int_0^H \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \sin\frac{p\pi x}{L} \sin\frac{q\pi y}{H}\, dy\, dx = \begin{cases} \frac{LH}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

41. $$\int_0^L \int_0^H \sin\frac{n\pi x}{L} \cos\frac{m\pi y}{H} \sin\frac{p\pi x}{L} \cos\frac{q\pi y}{H}\, dy\, dx = \begin{cases} \frac{LH}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{LH}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

42. $$\int_0^L \int_0^H \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H} \cos\frac{p\pi x}{L} \cos\frac{q\pi y}{H}\, dy\, dx = \begin{cases} LH, & \text{if } n = m = p = q = 0 \\ \frac{LH}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{LH}{2}, & \text{if } n = p = 0 \text{ and } m = q \neq 0 \\ \frac{LH}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

43. $$\int_0^a \int_0^L J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\overline{m}} r}{a})\, r \cos\frac{k\pi\theta}{L} \cos\frac{\overline{k}\pi\theta}{L}\, d\theta\, dr = \begin{cases} \frac{a^2 L}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k = 0 \\ \frac{a^2 L}{4} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

44. $$\int_0^a \int_0^L J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\overline{m}} r}{a})\, r \sin\frac{k\pi\theta}{L} \sin\frac{\overline{k}\pi\theta}{L}\, d\theta\, dr = \begin{cases} \frac{a^2 L}{4} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

45. $$\int_0^a \int_{-L}^L J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\overline{m}} r}{a})\, r \cos\frac{k\pi\theta}{L} \cos\frac{\overline{k}\pi\theta}{L}\, d\theta\, dr = \begin{cases} a^2 L J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k = 0 \\ \frac{a^2 L}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

46. $$\int_0^a \int_{-L}^L J_z(\tfrac{\alpha_m r}{a}) J_z(\tfrac{\alpha_{\overline{m}} r}{a})\, r \sin\tfrac{k\pi\theta}{L} \sin\tfrac{\overline{k}\pi\theta}{L}\, d\theta\, dr = \begin{cases} \frac{a^2 L}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

47*. $$\int_0^\pi \int_0^L P_n^m(\cos\phi) P_{\overline{n}}^m(\cos\phi) \sin\phi \cos\tfrac{k\pi\theta}{L} \cos\tfrac{\overline{k}\pi\theta}{L}\, d\theta\, d\phi = \begin{cases} \frac{2L}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \text{ and } \overline{k} = k = 0 \\ \frac{L}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

48*. $$\int_0^\pi \int_0^L P_n^m(\cos\phi) P_{\overline{n}}^m(\cos\phi) \sin\phi \sin\tfrac{k\pi\theta}{L} \sin\tfrac{\overline{k}\pi\theta}{L}\, d\theta\, d\phi = \begin{cases} \frac{L}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

49*. $$\int_0^\pi \int_{-L}^L P_n^m(\cos\phi) P_{\overline{n}}^m(\cos\phi) \sin\phi \cos\tfrac{k\pi\theta}{L} \cos\tfrac{\overline{k}\pi\theta}{L}\, d\theta\, d\phi = \begin{cases} \frac{4L}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \text{ and } \overline{k} = k = 0 \\ \frac{2L}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

50*. $$\int_0^\pi \int_{-L}^L P_n^m(\cos\phi) P_{\overline{n}}^m(\cos\phi) \sin\phi \sin\tfrac{k\pi\theta}{L} \sin\tfrac{\overline{k}\pi\theta}{L}\, d\theta\, d\phi = \begin{cases} \frac{2L}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

* It also holdss if $k = m$. For $m = 0$, $P_k^m(s) = P_k(s)$ and $\frac{(n+m)!}{(n-m)!} = 1$.

Definite Triple Integrals

Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

51**. $$\int_0^a \int_0^\pi \int_0^L J_z(\tfrac{\alpha_m \rho}{a}) J_z(\tfrac{\alpha_{\overline{m}} \rho}{a})\, \rho\, P_k^l(\cos\phi) P_{\overline{k}}^l(\cos\phi) \sin\phi \cos\tfrac{q\pi\theta}{L} \cos\tfrac{\overline{q}\pi\theta}{L}\, d\theta\, d\phi\, d\rho =$$
$$\begin{cases} \frac{a^2 L}{2k+1}\frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m,\ \overline{k} = k \geq l \text{ and } \overline{q} = q = 0 \\ \frac{a^2 L}{2(2k+1)}\frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m,\ \overline{k} = k \geq l \text{ and } \overline{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

52**. $$\int_0^a \int_0^\pi \int_0^L J_z(\tfrac{\alpha_m \rho}{a}) J_z(\tfrac{\alpha_{\overline{m}} \rho}{a})\, \rho\, P_k^l(\cos\phi) P_{\overline{k}}^l(\cos\phi) \sin\phi \sin\tfrac{q\pi\theta}{L} \sin\tfrac{\overline{q}\pi\theta}{L}\, d\theta\, d\phi\, d\rho =$$
$$\begin{cases} \frac{a^2 L}{2(2k+1)}\frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m,\ \overline{k} = k \geq l \text{ and } \overline{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

53**. $$\int_0^a \int_0^\pi \int_{-L}^L J_z(\tfrac{\alpha_m \rho}{a}) J_z(\tfrac{\alpha_{\overline{m}} \rho}{a})\, \rho\, P_k^l(\cos\phi) P_{\overline{k}}^l(\cos\phi) \sin\phi \cos\tfrac{q\pi\theta}{L} \cos\tfrac{\overline{q}\pi\theta}{L}\, d\theta\, d\phi\, d\rho =$$
$$\begin{cases} \frac{2a^2 L}{2k+1}\frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m,\ \overline{k} = k \geq l \text{ and } \overline{q} = q = 0 \\ \frac{a^2 L}{2k+1}\frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m,\ \overline{k} = k \geq l \text{ and } \overline{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

54**. $$\int_0^a \int_0^\pi \int_{-L}^L J_z(\tfrac{\alpha_m \rho}{a}) J_z(\tfrac{\alpha_{\overline{m}} \rho}{a})\, \rho\, P_k^l(\cos\phi) P_{\overline{k}}^l(\cos\phi) \sin\phi \sin\tfrac{q\pi\theta}{L} \sin\tfrac{\overline{q}\pi\theta}{L}\, d\theta\, d\phi\, d\rho =$$
$$\begin{cases} \frac{a^2 L}{2k+1}\frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m,\ \overline{k} = k \geq l \text{ and } \overline{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

** For $l = 0$, $P_n^l(s) = P_n(s)$ and $\frac{(k+l)!}{(k-1)!} = 1$.

Ordinary Differential Equations

55. First Order Linear ODE: $y' + f(x)y = g(x)$

Integrating Factor: $\mu(x) = e^{\int f(x)dx}$ with $C = 0\,,\ \mu(x)y'(x) + \mu(x)f(x)y(x) = \mu(x)g(x) \Longrightarrow$

$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)g(x) \Longrightarrow \mu(x)y(x) = \displaystyle\int \mu(x)g(x)\,dx + C \Longrightarrow y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dt + \frac{C}{\mu(x)}$

Or, integrating factor: $\mu(x) = e^{\int_{x_0}^{x} f(t)dt}$ and $y(x) = \frac{1}{\mu(x)}\displaystyle\int_{x_0}^{x} \mu(t)g(t)\,dt + \frac{y(x_0)}{\mu(x)}$

56. First Order Separable ODE: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

Implicit Solution: $\displaystyle\int h(y)\,dy = \int g(x)\,dx \Longrightarrow H(y) = G(x) + C$ with $H' = h$ and $G' = g$

Or, $\displaystyle\int_{y(x_0)}^{y} h(t)\,dt = \int_{x_0}^{x} g(t)\,dt$

57. Second Order Linear ODE with Constant Coefficients: $ay'' + by' + cy = 0$

Characteristics Equation: $ar^2 + br + c = 0$ with solutions r_1 and r_2

$$y(x) = \begin{cases} c_1 e^{r_1 x} + c_2 e^{r_2 x}, & \text{if } r_1 \text{ and } r_2 \text{ are real-valued and unequal} \\ c_1 e^{r_1 x} + c_2\, x\, e^{r_1 x}, & \text{if } r_1 = r_2 \\ c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x, & \text{if } r_1,\ r_2 = \alpha \pm \beta i \end{cases}$$

If $r_1,\ r_2 = \pm r$, then $y(x) = c_1 e^{-rx} + c_2 e^{rx}$ or $y(x) = c_1\cosh rx + c_2\sinh rx$ or

$y(x) = c_1\cosh r(x - x_0) + c_2\sinh r(x - x_0)$ or

$y(x) = c_1\sinh r(x - x_0) + c_2\sinh rx$ or $y(x) = c_1\cosh r(x - x_0) + c_2\cosh rx$

58. Second Order Linear Nonhomogeneous ODE: $y'' + p(x)y' + q(x)y = g(x)$

General Solution: $y(x) = y_h(x) + y_p(x)$ where the homogeneous solution $y_h(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, while y_1 and y_2 are two linearly independent solutions of the same homogeneous equation, and the particular solution $y_p(x)$ is a solution to the nonhomogeneous equation $y'' + p(x)y' + q(x)y = g(x)$.

Method of Variation of Parameter: $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where $u_1'(x) = \frac{-y_2(x)g(x)}{W(x)}$, $u_2'(x) = \frac{y_1(x)g(x)}{W(x)}$ and the Wronskian $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

$$y_p(x) = y_1(x)\int \frac{-y_2(x)g(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)g(x)}{W(x)}\,dx \text{ or}$$

$$y_p(x) = y_1(x)\int_{x_0}^{x} \frac{-y_2(t)g(t)}{W(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)g(t)}{W(t)}\,dt$$

59. Cauchy-Euler Equation: $ax^2y'' + bxy' + cy = 0$

Indicial (or Characteristic) Equation: $ar^2 + (b - a)r + c = 0$ or $ar(r-1) + br + c = 0$ with solutions r_1 and r_2

$$y(x) = \begin{cases} c_1|x|^{r_1} + c_2|x|^{r_2}, & \text{if } r_1 \text{ and } r_2 \text{ are real-valued and unequal} \\ (c_1 + c_2\ln|x|)|x|^{r_1}, & \text{if } r_1 = r_2 \\ |x|^{\alpha}[c_1\cos(\beta\ln|x|) + c_2\sin(\beta\ln|x|)], & \text{if } r_1,\ r_2 = \alpha \pm \beta i \end{cases}$$

60. $x^2\dfrac{d^2\phi}{dx^2} + x\dfrac{d\phi}{dx} - n^2\phi = 0$ and $\phi(0)$ bounded $\Longrightarrow \phi(x) = x^n$ for $n = 0\,,\,1\,,\cdots$

Rayleigh Quotients

61. $\dfrac{d}{dx}\left[p(x)\,\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0 \Longrightarrow \lambda = \dfrac{-p(x)\phi(x)\,\frac{d\phi}{dx}\Big|_a^b + \displaystyle\int_a^b \left[p(x)(\frac{d\phi}{dx})^2 - q(x)\phi^2(x)\right]\,dx}{\displaystyle\int_a^b \phi^2(x)\sigma(x)\,dx}$

62. $\nabla^2\phi + \lambda\phi = 0 \Longrightarrow \lambda = \dfrac{-\displaystyle\oint_C \phi\nabla\phi\cdot\hat{n}\,ds + \iint_R |\nabla\phi|^2\,dA}{\displaystyle\iint_R \phi^2\,dA}$

Lagrange's Identity and Green's Formula

For $L(\phi) = \dfrac{d}{dx}\left[p(x)\,\dfrac{d\phi}{dx}\right] + q(x)\phi,$

63. $uL(v) - vL(u) = \dfrac{d}{dx}\left[p(x)\left(u(x)\dfrac{dv}{dx} - v(x)\dfrac{du}{dx}\right)\right]$

64. $\displaystyle\int_a^b [uL(v) - vL(u)]\,dx = p(x)\,\left[u(x)\frac{dv}{dx} - v(x)\frac{du}{dx}\right]\Bigg|_a^b$

Green's Identities

65. $\displaystyle\iint_R u\nabla^2 v\,dA = \oint_C u\nabla v\cdot\hat{n}\,ds - \iint_R \nabla u\cdot\nabla v\,dA$

66. $\displaystyle\iint_R (u\nabla^2 v - v\nabla^2 u)\,dA = \oint_C (u\nabla v - v\nabla u)\cdot\hat{n}\,ds$

67. $\displaystyle\iiint_\Omega (u\nabla^2 v - v\nabla^2 u)\,dV = \oiint_{\partial\Omega} (u\nabla v - v\nabla u)\cdot\hat{n}\,dS$

Eigenvalue Problems

68. $\dfrac{d^2\phi}{dx^2} = -\lambda\phi\,,\ \phi(0) = 0$ and $\phi(L) = 0 \Longrightarrow \lambda = (\frac{n\pi}{L})^2\,,\ \phi(x) = \sin\frac{n\pi x}{L}$ for $n = 1\,,\,2\,,\cdots$

69. $\dfrac{d^2\phi}{dx^2} = -\lambda\phi\,,\ \dfrac{d\phi}{dx}(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0 \Longrightarrow \lambda = (\frac{n\pi}{L})^2\,,\ \phi(x) = \cos\frac{n\pi x}{L}$ for $n = 0\,,\,1\,,\cdots$

70. $\left\{\begin{array}{l} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(-L) = \phi(L) \\ \dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L) \end{array}\right. \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 \\ \phi(x) = \cos\frac{n\pi x}{L} \text{ and } \sin\frac{n\pi x}{L} \end{array}$ for $n = 0\,,\,1\,,\cdots$

71. $\left\{\begin{array}{l} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = 0 \\ \dfrac{d\phi}{dx}(L) = 0 \end{array}\right. \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2L}]^2 \\ \phi(x) = \sin\frac{(2n-1)\pi x}{2L} \end{array}$ for $n = 1\,,\,2\,,\cdots$

72. $$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \dfrac{d\phi}{dx}(0) = 0 \\ \phi(L) = 0 \end{cases} \implies \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2L}]^2 \\ \phi(x) = \cos\frac{(2n-1)\pi x}{2L} \end{array} \quad \text{for } n = 1\,,\,2\,,\cdots$$

73. $$\begin{cases} x^2\dfrac{d^2\phi}{dx^2} + x\dfrac{d\phi}{dx} + (\lambda x^2 - n^2)\phi = 0 \\ \phi(0) \text{ bounded} \\ \phi(a) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{\alpha_m}{a})^2 \\ \phi(x) = J_n(\frac{\alpha_m x}{a}) \end{array} \quad \text{for } 0 < \alpha_1 < \alpha_2 < \cdots \text{ zeros of } J_n(x)$$

74. $$\begin{cases} \dfrac{d}{d\rho}\left[\rho^2\dfrac{df}{d\rho}\right] + [\lambda\rho^2 - n(n+1)]f(\rho) = 0 \\ f(0) \text{ bounded} \\ f(a) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{\alpha_k}{a})^2 \\ f(\rho) = \rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\frac{\alpha_k\rho}{a}) \\ \text{for } 0 < \alpha_1 < \alpha_2 < \cdots \text{ zeros of } J_{n+\frac{1}{2}}(\rho) \end{array}$$

75. $$\begin{cases} x^2\dfrac{d^2\phi}{dx^2} + x\dfrac{d\phi}{dx} + (\lambda x^2 - n^2)\phi = 0 \\ \phi(0) \text{ bounded} \\ \dfrac{d\phi}{dx}(a) = 0 \end{cases} \implies \begin{array}{l} n = 0\,,\ \lambda = 0\,,\ \phi(x) = 1 \\ n > 0\,,\ \lambda = (\frac{\beta_m}{a})^2\,,\ \phi(x) = J_n(\frac{\beta_m x}{a}) \\ \text{for } 0 < \beta_1 < \beta_2 < \cdots \text{ zeros of } J_n'(x) \end{array}$$

76*. $$\begin{cases} \dfrac{d}{d\phi}\left[\sin\phi\dfrac{dg}{d\phi}\right] + \left(\mu\sin\phi - \dfrac{m^2}{\sin\phi}\right)g(\phi) = 0 \\ g(0) \text{ and } g(\pi) \text{ bounded} \end{cases} \implies \begin{array}{l} \mu = n(n+1) \\ g(\phi) = P_n^m(\cos\phi) \end{array} \quad \text{for } n = m\,,\,m+1\,,\cdots$$

77*. $$\begin{cases} (1-s^2)\dfrac{d^2\phi}{ds^2} - 2s\dfrac{d\phi}{ds} + \left(\mu - \frac{m^2}{1-s^2}\right)\phi = 0 \\ \phi(-1) \text{ and } \phi(1) \text{ bounded} \end{cases} \implies \begin{array}{l} \mu = n(n+1) \\ \phi(s) = P_n^m(s) \end{array} \quad \text{for } n = m\,,\,m+1\,,\cdots$$

* For $m = 0$, $P_n^m(s) = P_n(s)$.

Two-Dimensional Eigenvalue Problems

78. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \phi(L,\,y) = 0 \\ \phi(x,\,0) = \phi(x,\,H) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2 \\ \phi(x) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \end{array} \quad \text{for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

79. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \dfrac{\partial\phi}{\partial x}(L,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \dfrac{\partial\phi}{\partial y}(x,\,H) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2 \\ \phi(x) = \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H} \end{array} \quad \text{for } n = 0\,,\,1\,,\cdots \text{ and } m = 0\,,\,1\,,\cdots$$

80. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \phi(L,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \dfrac{\partial\phi}{\partial y}(x,\,H) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2 \\ \phi(x) = \sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H} \end{array} \quad \text{for } n = 1\,,\,2\,,\cdots \text{ and } m = 0\,,\,1\,,\cdots$$

81. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \dfrac{\partial\phi}{\partial x}(L,\,0) = 0 \\ \phi(x,\,0) = \phi(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2 \\ \phi(x) = \cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \end{array} \text{ for } n = 0\,,\,1\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

82. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \phi(L,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \phi(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + [\frac{(2m-1)\pi}{2H}]^2 \\ \phi(x) = \sin\frac{n\pi x}{L}\cos\frac{(2m-1)\pi y}{2H} \end{array} \text{ for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

83. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \phi(L,\,y) = 0 \\ \phi(x,\,0) = \frac{\partial\phi}{\partial y}(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + [\frac{(2m-1)\pi}{2H}]^2 \\ \phi(x) = \sin\frac{n\pi x}{L}\sin\frac{(2m-1)\pi y}{2H} \end{array} \text{ for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

84. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \dfrac{\partial\phi}{\partial x}(L,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \phi(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + [\frac{(2m-1)\pi}{2H}]^2 \\ \phi(x) = \cos\frac{n\pi x}{L}\cos\frac{(2m-1)\pi y}{2H} \end{array} \text{ for } n = 0\,,\,1\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

85. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \dfrac{\partial\phi}{\partial x}(L,\,y) = 0 \\ \phi(x,\,0) = \dfrac{\partial\phi}{\partial y}(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{L})^2 + [\frac{(2m-1)\pi}{2H}]^2 \\ \phi(x) = \cos\frac{n\pi x}{L}\sin\frac{(2m-1)\pi y}{2H} \end{array} \text{ for } n = 0\,,\,1\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

86. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \phi(L,\,y) = 0 \\ \phi(x,\,0) = \dfrac{\partial\phi}{\partial y}(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2L}]^2 + [\frac{(2m-1)\pi}{2H}]^2 \\ \phi(x) = \cos\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi y}{2H} \end{array} \text{ for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

87. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \phi(L,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \phi(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2L}]^2 + [\frac{(2m-1)\pi}{2H}]^2 \\ \phi(x) = \cos\frac{(2n-1)\pi x}{2L}\cos\frac{(2m-1)\pi y}{2H} \end{array} \text{ for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

88. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \dfrac{\partial\phi}{\partial x}(L,\,y) = 0 \\ \phi(x,\,0) = \dfrac{\partial\phi}{\partial y}(x,\,H) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2L}]^2 + [\frac{(2m-1)\pi}{2H}]^2 \\ \phi(x) = \sin\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi y}{2H} \end{array} \text{ for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

Supporting Theorems

89. Green's Theorem (vector version)

Let R be a region in $\Re^2$ bounded by a piecewise-smooth, simple closed curve C with counterclockwise orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containing R, then $\displaystyle\iint\limits_R \nabla \cdot \vec{F}\, dA = \oint\limits_C \vec{F} \cdot \hat{n}\, ds\,.$

90. Divergence Theorem

Let Ω be a simple solid region in $\Re^3$ and let $\partial\Omega$ be its boundary with the outward orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containing Ω, then $\displaystyle\iiint\limits_\Omega \nabla \cdot \vec{F}\, dV = \oiint\limits_{\partial\Omega} \vec{F} \cdot \hat{n}\, dS\,.$

91. If function f is continuous, $f(x) \not\equiv 0$ and $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\, dx > 0\,.$

92. For a continuous nonnegative function f if $\displaystyle\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ for $a \leq x \leq b\,.$

93. Uniform Convergence Definition

The sequence of functions $f_n : D \longrightarrow \Re\,,\ n = 1\,,\ 2\,, \cdots$, is said to converge uniformly to the function $f : D \longrightarrow \Re$ if for every $\epsilon > 0$, there is a natural number N such that for all $x \in D$ we have $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$.

94. Weierstrass M Test (A test for uniform convergence.)

Suppose for each function $f_n : D \longrightarrow \Re\,,\ n = 1\,,\ 2\,, \cdots$, there exists a constant M_n with $|f_n(x)| \leq M_n$ for all $x \in D$, and $\displaystyle\sum_{n=1}^{\infty} M_n$ converges. Then $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly.

95. Interchanging Limit and Integral

Suppose functions $f_n : [a,\, b] \longrightarrow \Re\,,\ n = 1\,,\ 2\,, \cdots$, are continuous and converge uniformly to a function $f : [a,\, b] \longrightarrow \Re$. Then $\displaystyle\lim_{n\to\infty} \left[\int_a^b f_n(x)\, dx \right] = \int_a^b \left[\lim_{n\to\infty} f_n(x) \right] dx = \int_a^b f(x)\, dx\,.$

96. Interchanging Integral and Summation

Suppose functions $f_n : [a,\, b] \longrightarrow \Re\,,\ n = 1\,,\ 2\,, \cdots$, are continuous, and $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly.

Then $\displaystyle\int_a^b \left[\sum_{n=1}^{\infty} f_n(x) \right] dx = \sum_{n=1}^{\infty} \left[\int_a^b f(x)\, dx \right].$

97. Interchanging Differentiation and Summation

Suppose functions $f_n\,,\ n = 1\,,\ 2\,, \cdots$, are continuously differentiable, $\displaystyle\sum_{n=1}^{\infty} f_n$ converges pointwise, and $\displaystyle\sum_{n=1}^{\infty} f_n'$ converges uniformly. Then $\displaystyle\frac{d}{dx}\left[\sum_{n=1}^{\infty} f_n(x) \right] = \sum_{n=1}^{\infty} \left[\frac{d}{dx} f_n(x) \right].$

98. Leibniz Integral Rule (Interchanging differentiation and integration with respect to different variables.)

Suppose functions $f(x,\, y)$ and $\dfrac{\partial f}{\partial y}(x,\, y)$ are continuous on $[a,\, b] \times [c,\, d]$. Then

$$\frac{d}{dy}\left[\int_a^b f(x,\, y)\, dx\right] = \int_a^b \left[\frac{\partial}{\partial y} f(x,\, y)\right] dx.$$

Fourier Series

99. If $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \dfrac{n\pi x}{L}$ for $0 < x < L$, then $A_0 = \frac{1}{L}\int_0^L f(x)\, dx$ and $A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$

100. If $f(x) = \sum_{n=1}^{\infty} B_n \sin \dfrac{n\pi x}{L}$ for $0 < x < L$, then $B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$

101. If $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \dfrac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{L}$ for $-L < x < L$, then $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$,

$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$

102. If $f(x) = \sum_{n=1}^{\infty} A_n \cos \dfrac{(2n-1)\pi x}{2L}$ for $0 < x < L$, then $A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L}\, dx$

103. If $f(x) = \sum_{n=1}^{\infty} B_n \sin \dfrac{(2n-1)\pi x}{2L}$ for $0 < x < L$, then $B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$

Generalized Fourier Series

104. Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

If $f(r) = \sum_{m=1}^{\infty} a_m J_z(\dfrac{\alpha_m r}{a})$ for $0 < r < a$, then $a_m = \dfrac{\int_0^a f(r) J_z(\frac{\alpha_m r}{a}) r\, dr}{\int_0^a J_z^2(\frac{\alpha_m r}{a}) r\, dr} = \dfrac{2\int_0^a f(r) J_z(\frac{\alpha_m r}{a}) r\, dr}{a^2 J_{z+1}^2(\alpha_m)}$

105. Suppose $0 < \beta_1 < \beta_2 < \cdots$ are zeros of $J_0'(x)$.

If $f(r) = a_0 + \sum_{m=1}^{\infty} a_m J_0(\dfrac{\alpha_m r}{a})$ for $0 < r < a$, then $a_0 = \frac{2}{a^2}\int_0^a f(r)\, r\, dr$ and

$$a_m = \frac{\int_0^a f(r) J_0(\frac{\beta_m r}{a}) r\, dr}{\int_0^a J_0^2(\frac{\beta_m r}{a}) r\, dr} = \frac{2\int_0^a f(r) J_0(\frac{\beta_m r}{a}) r\, dr}{a^2 J_0^2(\beta_m)}$$

106. Suppose $0 < \beta_1 < \beta_2 < \cdots$ are zeros of $J_z'(x)$ and $z > 0$.

If $f(r) = \sum_{m=1}^{\infty} a_m J_z(\dfrac{\alpha_m r}{a})$ for $0 < r < a$, then

$$a_m = \frac{\int_0^a f(r) J_z(\frac{\beta_m r}{a}) r\, dr}{\int_0^a J_z^2(\frac{\beta_m r}{a}) r\, dr} = \frac{2\int_0^a f(r) J_z(\frac{\beta_m r}{a}) r\, dr}{a^2[J_z^2(\beta_m) - J_{z-1}(\beta_m) J_{z+1}(\beta_m)]}$$

107*. If $f(\phi) = \sum_{n=m}^{\infty} a_n P_n^m(\cos\phi)$ for $0 < \phi < \pi$, then

$$a_n = \frac{\int_0^{\pi} f(\phi) P_n^m(\cos\phi)\sin\phi\, d\phi}{\int_0^{\pi} [P_n^m(\cos\phi)]^2 \sin\phi\, d\phi} = \tfrac{2n+1}{2}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi} f(\phi) P_n^m(\cos\phi)\sin\phi\, d\phi$$

108*. If $f(x) = \sum_{n=m}^{\infty} a_n P_n^m(x)$ for $-1 < x < 1$, then

$$a_n = \frac{\int_{-1}^{1} f(x) P_n^m(x)\, dx}{\int_{-1}^{1} [P_n^m(x)]^2\, dx} = \tfrac{2n+1}{2}\tfrac{(n-m)!}{(n+m)!}\int_{-1}^{1} f(x) P_n^m(x)\, dx$$

* For $m = 0$, $P_n^m(s) = P_n(s)$ and $\frac{(n-m)!}{(n+m)!} = 1$.

Double Fourier Series

109. If $f(x,\, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$ for $(x,\, y) \in (0,\, L) \times (0,\, H)$, then

$$B_{nm} = \tfrac{4}{LH}\int_0^L \int_0^H f(x,\, y) \sin\tfrac{n\pi x}{L} \sin\tfrac{m\pi y}{H}\, dy\, dx$$

110. If $f(x,\, y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}$ for $(x,\, y) \in (0,\, L) \times (0,\, H)$, then

$$A_{00} = \tfrac{1}{LH}\int_0^L \int_0^H f(x,\, y)\, dy\, dx\, ,\ A_{n0} = \tfrac{2}{LH}\int_0^L \int_0^H f(x,\, y) \cos\tfrac{n\pi x}{L}\, dy\, dx\, ,$$

$$A_{0m} = \tfrac{2}{LH}\int_0^L \int_0^H f(x,\, y) \cos\tfrac{m\pi y}{L}\, dy\, dx \text{ and } A_{nm} = \tfrac{4}{LH}\int_0^L \int_0^H f(x,\, y) \cos\tfrac{n\pi x}{L} \cos\tfrac{m\pi y}{H}\, dy\, dx$$

111. If $f(x,\, y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm} \sin\frac{n\pi x}{L} \cos\frac{m\pi y}{H}$ for $(x,\, y) \in (0,\, L) \times (0,\, H)$, then

$$C_{n0} = \tfrac{2}{LH}\int_0^L \int_0^H f(x,\, y) \sin\tfrac{n\pi x}{L}\, dy\, dx \text{ and } C_{nm} = \tfrac{4}{LH}\int_0^L \int_0^H f(x,\, y) \sin\tfrac{n\pi x}{L} \cos\tfrac{m\pi y}{H}\, dy\, dx$$

112. If $f(x,\, y) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\frac{m\pi x}{L} \cos\frac{n\pi y}{H} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm} \sin\frac{m\pi x}{L} \cos\frac{n\pi y}{H}$ for

$(x,\, y) \in (0,\, L) \times (-H,\, H)$, then $C_{0m} = \frac{1}{LH}\int_0^L \int_{-H}^H f(x,\, y) \sin\frac{m\pi x}{L}\, dy\, dx$,

$$C_{nm} = \tfrac{2}{LH}\int_0^L \int_{-H}^H f(x,\, y) \sin\tfrac{m\pi x}{L} \cos\tfrac{n\pi y}{H}\, dy\, dx \text{ and } D_{nm} = \tfrac{2}{LH}\int_0^L \int_{-H}^H f(x,\, y) \sin\tfrac{m\pi x}{L} \sin\tfrac{n\pi y}{H}\, dy\, dx$$

113. If $f(x,\, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \cos\frac{(2n-1)\pi x}{2L} \cos\frac{(2m-1)\pi y}{2H}$ for $(x,\, y) \in (0,\, L) \times (0,\, H)$, then

$$C_{nm} = \tfrac{4}{LH}\int_0^L \int_0^H f(x,\, y) \cos\tfrac{(2n-1)\pi x}{2L} \cos\tfrac{(2m-1)\pi y}{2H}\, dy\, dx$$

114. If $f(x,\,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\sin\dfrac{n\pi x}{L}\cos\dfrac{(2m-1)\pi y}{2H}$ for $(x,\,y)\in(0,\,L)\times(0,\,H)$, then

$$C_{nm}=\tfrac{4}{LH}\int_0^L\int_0^H f(x,\,y)\sin\tfrac{n\pi x}{L}\cos\tfrac{(2m-1)\pi y}{2H}\,dy\,dx$$

115. If $f(x,\,y)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{nm}\cos\dfrac{n\pi x}{L}\cos\dfrac{(2m-1)\pi y}{2H}$ for $(x,\,y)\in(0,\,L)\times(0,\,H)$, then

$$C_{0m}=\tfrac{2}{LH}\int_0^H f(x,\,y)\cos\tfrac{(2m-1)\pi y}{2H}\,dy \text{ and } C_{nm}=\tfrac{4}{LH}\int_0^L\int_0^H f(x,\,y)\cos\tfrac{n\pi x}{L}\cos\tfrac{(2m-1)\pi y}{2H}\,dy\,dx$$

116. If $f(x,\,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\sin\dfrac{n\pi x}{L}\sin\dfrac{(2m-1)\pi y}{2H}$ for $(x,\,y)\in(0,\,L)\times(0,\,H)$, then

$$C_{nm}=\tfrac{4}{LH}\int_0^L\int_0^H f(x,\,y)\sin\tfrac{n\pi x}{L}\sin\tfrac{(2m-1)\pi y}{2H}\,dy\,dx$$

117. If $f(x,\,y)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{nm}\cos\dfrac{n\pi x}{L}\sin\dfrac{(2m-1)\pi y}{2H}$ for $(x,\,y)\in(0,\,L)\times(0,\,H)$, then

$$C_{0m}=\tfrac{2}{LH}\int_0^H f(x,\,y)\sin\tfrac{(2m-1)\pi y}{2H}\,dy \text{ and } C_{nm}=\tfrac{4}{LH}\int_0^L\int_0^H f(x,\,y)\cos\tfrac{n\pi x}{L}\sin\tfrac{(2m-1)\pi y}{2H}\,dy\,dx$$

118. If $f(x,\,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\sin\dfrac{(2n-1)\pi x}{L}\sin\dfrac{(2m-1)\pi y}{2H}$ for $(x,\,y)\in(0,\,L)\times(0,\,H)$, then

$$C_{nm}=\tfrac{4}{LH}\int_0^L\int_0^H f(x,\,y)\sin\tfrac{(2n-1)\pi x}{L}\sin\tfrac{(2m-1)\pi y}{2H}\,dy\,dx$$

Generalized Double Fourier Series

Suppose $0<\alpha_1<\alpha_2<\cdots$ are zeros of $J_z(x)$.

119. If $f(r,\,\theta)=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}A_{mk}J_z(\dfrac{\alpha_m r}{a})\cos\dfrac{k\pi\theta}{L}$ for $(r,\,\theta)\in(0,\,a)\times(0,\,L)$, then

$$A_{m0}=\tfrac{2}{a^2LJ_{z+1}^2(\alpha_m)}\int_0^a\int_0^L f(r,\,\theta)J_z(\tfrac{\alpha_m r}{a})r\,d\theta\,dr \text{ and}$$

$$A_{mk}=\tfrac{4}{a^2LJ_{z+1}^2(\alpha_m)}\int_0^a\int_0^L f(r,\,\theta)\cos\tfrac{k\pi\theta}{L}J_z(\tfrac{\alpha_m r}{a})r\,d\theta\,dr$$

120. If $f(r,\,\theta)=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}B_{mk}J_z(\dfrac{\alpha_m r}{a})\sin\dfrac{k\pi\theta}{L}$ for $(r,\,\theta)\in(0,\,a)\times(0,\,L)$, then

$$B_{mk}=\tfrac{4}{a^2LJ_{z+1}^2(\alpha_m)}\int_0^a\int_0^L f(r,\,\theta)\sin\tfrac{k\pi\theta}{L}J_z(\tfrac{\alpha_m r}{a})r\,d\theta\,dr$$

121. If $f(r,\,\theta)=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}A_{mk}J_z(\dfrac{\alpha_m r}{a})\cos\dfrac{k\pi\theta}{L}+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}B_{mk}J_z(\dfrac{\alpha_m r}{a})\sin\dfrac{k\pi\theta}{L}$ for

$(r,\,\theta)\in(0,\,a)\times(-L,\,L)$, then $A_{m0}=\tfrac{1}{a^2LJ_{z+1}^2(\alpha_m)}\int_0^a\int_{-L}^L f(r,\,\theta)J_z(\tfrac{\alpha_m r}{a})r\,d\theta\,dr\,,$

$$A_{mk}=\tfrac{2}{a^2LJ_{z+1}^2(\alpha_m)}\int_0^a\int_{-L}^L f(r,\,\theta)\cos\tfrac{k\pi\theta}{L}J_z(\tfrac{\alpha_m r}{a})r\,d\theta\,dr \text{ and}$$

$$B_{mk}=\tfrac{2}{a^2LJ_{z+1}^2(\alpha_m)}\int_0^a\int_{-L}^L f(r,\,\theta)\sin\tfrac{k\pi\theta}{L}J_z(\tfrac{\alpha_m r}{a})r\,d\theta\,dr$$

122*. If $f(\theta,\,\phi)=\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}A_{nk}P_n^m(\cos\phi)\cos\dfrac{k\pi\theta}{L}$ for $(\theta,\,\phi)\in(0,\,L)\times(0,\,\pi)$, then

$$A_{n0}=\tfrac{2n+1}{2L}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{L}f(\theta,\,\phi)P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi \text{ and}$$

$$A_{nk}=\tfrac{2n+1}{L}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{L}f(\theta,\,\phi)\cos\tfrac{k\pi\theta}{L}P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi$$

123*. If $f(\theta,\,\phi)=\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}A_{nk}P_n^m(\cos\phi)\sin\dfrac{k\pi\theta}{L}$ for $(\theta,\,\phi)\in(0,\,L)\times(0,\,\pi)$, then

$$B_{nk}=\tfrac{2n+1}{L}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{L}f(\theta,\,\phi)\sin\tfrac{k\pi\theta}{L}P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi$$

124*. If $f(\theta,\,\phi)=\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}A_{nk}P_n^m(\cos\phi)\cos\dfrac{k\pi\theta}{L}+\sum_{k=1}^{\infty}\sum_{n=m}^{\infty}B_{nk}P_n^m(\cos\phi)\sin\dfrac{k\pi\theta}{L}$ for

$(\theta,\,\phi)\in(-L,\,L)\times(0,\,\pi)$, then $A_{n0}=\tfrac{2n+1}{4L}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_{-L}^{L}f(\theta,\,\phi)P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi\,,$

$$A_{nk}=\tfrac{2n+1}{2L}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_{-L}^{L}f(\theta,\,\phi)\cos\tfrac{k\pi\theta}{L}P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi \text{ and}$$

$$B_{nk}=\tfrac{2n+1}{2L}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_{-L}^{L}f(\theta,\,\phi)\sin\tfrac{k\pi\theta}{L}P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi$$

* It also holds if $k=m$. For $m=0$, $P_n^m(s)=P_n(s)$ and $\tfrac{(n-m)!}{(n+m)!}=1$.

Generalized Triple Fourier Series

Suppose $0<\alpha_1<\alpha_2<\cdots$ are zeros of $J_z(x)$.

125**. If $f(\rho,\,\theta,\,\phi)=\sum_{q=0}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}A_{mkq}J_z(\dfrac{\alpha_m\rho}{a})P_k^l(\cos\phi)\cos\dfrac{q\pi\theta}{L}$ for $(\rho,\,\theta,\,\phi)\in(0,\,a)\times(0,\,L)\times(0,\,\pi)$,

then $A_{mk0}=\tfrac{2k+1}{a^2LJ_{z+1}^2(\alpha_m)}\tfrac{(k-l)!}{(k+l)!}\int_0^{a}\int_0^{\pi}\int_0^{L}f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\,d\theta\,d\phi\,d\rho$ and

$$A_{mkq}=\tfrac{2(2k+1)}{a^2LJ_{z+1}^2(\alpha_m)}\tfrac{(k-l)!}{(k+l)!}\int_0^{a}\int_0^{\pi}\int_0^{L}f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\cos\tfrac{q\pi\theta}{L}\,d\theta\,d\phi\,d\rho$$

126**. If $f(\rho,\,\theta,\,\phi)=\sum_{q=1}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}B_{mkq}J_z(\dfrac{\alpha_m\rho}{a})P_k^l(\cos\phi)\sin\dfrac{q\pi\theta}{L}$ for $(\rho,\,\theta,\,\phi)\in(0,\,a)\times(0,\,L)\times(0,\,\pi)$,

then $B_{mkq}=\tfrac{2(2k+1)}{a^2LJ_{z+1}^2(\alpha_m)}\tfrac{(k-l)!}{(k+l)!}\int_0^{a}\int_0^{\pi}\int_0^{L}f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\sin\tfrac{q\pi\theta}{L}\,d\theta\,d\phi\,d\rho$

** For $l=0$, $P_k^l(s)=P_k(s)$ and $\tfrac{(k-l)!}{(k+l)!}=1$.

Suppose $0<\alpha_1<\alpha_2<\cdots$ are zeros of $J_z(x)$.

127*. If $f(\rho,\,\theta,\,\phi)=\sum_{q=0}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}A_{mkq}J_z(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\cos\frac{q\pi\theta}{L}+\sum_{q=1}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}B_{mkq}J_z(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\sin\frac{q\pi\theta}{L}$

for $(\rho,\,\theta,\,\phi)\in(0,\,a)\times(-L,\,L)\times(0,\,\pi)$, then

$$A_{mk0}=\frac{2k+1}{2a^2LJ_{z+1}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^\pi\int_{-L}^{L}f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\,d\theta\,d\phi\,d\rho\,,$$

$$A_{mkq}=\frac{2k+1}{a^2LJ_{z+1}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^\pi\int_{-L}^{L}f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\cos\tfrac{q\pi\theta}{L}\,d\theta\,d\phi\,d\rho \text{ and}$$

$$B_{mkq}=\frac{2k+1}{a^2LJ_{z+1}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^\pi\int_{-L}^{L}f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\sin\tfrac{q\pi\theta}{L}\,d\theta\,d\phi\,d\rho$$

128*. If $f(\rho,\,\theta,\,\phi)=\sum_{q=0}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}A_{mkq}\,\rho^{-\frac12}J_{z+\frac12}(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\cos\frac{q\pi\theta}{L}+$

$$\sum_{q=1}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}B_{mkq}\,\rho^{-\frac12}J_{z+\frac12}(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\sin\frac{q\pi\theta}{L}$$

for $(\rho,\,\theta,\,\phi)\in(0,\,a)\times(-L,\,L)\times(0,\,\pi)$, then

$$A_{mk0}=\frac{2k+1}{2a^2LJ_{z+\frac32}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^\pi\int_{-L}^{L}f(\rho,\,\theta,\,\phi)J_{z+\frac12}(\tfrac{\alpha_m\rho}{a})\rho^{\frac32}P_k^l(\cos\phi)\sin\phi\,d\theta\,d\phi\,d\rho\,,$$

$$A_{mkq}=\frac{2k+1}{a^2LJ_{z+\frac32}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^\pi\int_{-L}^{L}f(\rho,\,\theta,\,\phi)J_{z+\frac12}(\tfrac{\alpha_m\rho}{a})\rho^{\frac32}P_k^l(\cos\phi)\sin\phi\cos\tfrac{q\pi\theta}{L}\,d\theta\,d\phi\,d\rho \text{ and}$$

$$B_{mkq}=\frac{2k+1}{a^2LJ_{z+\frac32}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^\pi\int_{-L}^{L}f(\rho,\,\theta,\,\phi)J_{z+\frac12}(\tfrac{\alpha_m\rho}{a})\rho^{\frac32}P_k^l(\cos\phi)\sin\phi\sin\tfrac{q\pi\theta}{L}\,d\theta\,d\phi\,d\rho$$

* For $l=0$, $P_k^l(s)=P_k(s)$ and $\frac{(k-l)!}{(k+l)!}=1$

Table of Fourier Transforms

$f(x)=\mathcal{F}^{-1}\{F(w)\}=\int_{-\infty}^{\infty}F(\omega)e^{-i\omega x}\,d\omega$ $\qquad F(w)=\mathcal{F}^{-1}\{F(w)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{i\omega x}\,dx$

$f(x)=\mathcal{F}^{-1}\{F(w)\}$	$F(w)=\mathcal{F}\{f(x)\}$	$f(x)=\mathcal{F}^{-1}\{F(w)\}$	$F(w)=\mathcal{F}\{f(x)\}$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}\,e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}}\,e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\,\omega^2}$
$\frac{\partial}{\partial t}f(x,\,t)$	$\frac{\partial}{\partial t}F(\omega,\,t)$	$\frac{\partial f}{\partial x}$	$-i\,\omega\,F(\omega)$
$\frac{\partial^2 f}{\partial x^2}$	$(-i\,\omega)^2\,F(\omega)$	$\frac{\partial^n f}{\partial x^n}$	$(-i\,\omega)^n\,F(\omega)$
$\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\overline{x})g(x-\overline{x})\,d\overline{x}$	$F(\omega)G(\omega)\,,$ $G(w)=\mathcal{F}\{g(x)\}$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x-\overline{x})g(x)\,d\overline{x}$	$F(\omega)G(\omega)\,,$ $G(w)=\mathcal{F}\{g(x)\}$
$\delta(x-x_0)$	$\frac{1}{2\pi}\,e^{i\,\omega\,x_0}$	$f(x-\beta)$	$e^{i\,\omega\,\beta}F(\omega)$
$xf(x)$	$-i\,\frac{dF}{d\omega}$	$\frac{2\alpha}{x^2+\alpha^2}$	$e^{-\lvert\omega\rvert\,\alpha}$
$f(x)=\begin{cases}1, & \lvert x\rvert<a\\ 0, & \lvert x\rvert>a\end{cases}$	$\frac{1}{\pi}\frac{\sin a\omega}{\omega}$		

Table of Fourier Sine Transforms		Table of Fourier Cosine Transforms	
$f(x) = \mathcal{S}^{-1}\{F_S(w)\} = \int_0^\infty F_S(\omega)\sin\omega x\,d\omega$		$f(x) = \mathcal{C}^{-1}\{F_C(w)\} = \int_0^\infty F_C(\omega)\cos\omega x\,d\omega$	
$F_S(w) = \mathcal{S}\{f(x)\} = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$		$F_C(w) = \mathcal{C}\{f(x)\} = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$	
$f(x) = \mathcal{S}^{-1}\{F(w)\}$	$F_S(w) = \mathcal{S}\{f(x)\}$	$f(x) = \mathcal{C}^{-1}\{F_C(w)\}$	$F_C(w) = \mathcal{C}\{f(x)\}$
1	$\frac{2}{\pi}\frac{1}{\omega}$	$e^{-\alpha x^2}$	$\frac{2}{\sqrt{4\pi\alpha}}\,e^{-\frac{\omega^2}{4\alpha}}$
$e^{-\epsilon x}$	$\frac{2}{\pi}\frac{\omega}{\epsilon^2+\omega^2}$	$e^{-\epsilon x}$	$\frac{2}{\pi}\frac{\epsilon}{\epsilon^2+\omega^2}$
$\frac{\partial}{\partial t}f(x,\,t)$	$\frac{\partial}{\partial t}F_S(\omega,\,t)$	$\frac{\partial}{\partial t}f(x,\,t)$	$\frac{\partial}{\partial t}F_C(\omega,\,t)$
$\frac{\partial f}{\partial x}$	$-\omega\,\mathcal{C}\{f(x)\}$ $= -\omega\,F_C(\omega)$	$\frac{\partial f}{\partial x}$	$-\frac{2}{\pi}f(0) + \omega\,\mathcal{S}\{f(x)\}$ $= -\frac{2}{\pi}f(0) + \omega\,F_S(\omega)$
$\frac{\partial^2 f}{\partial x^2}$	$\frac{2}{\pi}\omega f(0) - \omega^2\,\mathcal{S}\{f(x)\}$ $= \frac{2}{\pi}\omega f(0) - \omega^2\,F_S(\omega)$	$\frac{\partial^2 f}{\partial x^2}$	$-\frac{2}{\pi}\omega\frac{\partial f}{\partial x}(0) - \omega^2\,\mathcal{C}\{f(x)\}$ $= -\frac{2}{\pi}\omega\frac{\partial f}{\partial x}(0) - \omega^2\,F_C(\omega)$
$\frac{x}{x^2+\beta^2}$	$e^{-\omega\beta}$	$\frac{\beta}{x^2+\beta^2}$	$e^{-\omega\beta}$
For an even function g, $\frac{1}{\pi}\int_0^\infty f(\overline{x})[g(x-\overline{x}) - g(x+\overline{x})]\,d\overline{x}$	$S\{f(x)\}\,C\{g(x)\}$ $= F_S(\omega)G_C(\omega)$	For an even function f, $\frac{1}{\pi}\int_0^\infty g(\overline{x})[f(x-\overline{x}) + f(x+\overline{x})]\,d\overline{x}$	$C\{f(x)\}\,C\{g(x)\}$ $= F_C(\omega)G_C(\omega)$
For an odd function f, $\frac{1}{\pi}\int_0^\infty g(\overline{x})[f(x+\overline{x}) + f(x-\overline{x})]\,d\overline{x}$	$S\{f(x)\}\,C\{g(x)\}$ $= F_S(\omega)G_C(\omega)$	For an even function g, $\frac{1}{\pi}\int_0^\infty f(\overline{x})[g(x-\overline{x}) + g(x+\overline{x})]\,d\overline{x}$	$C\{f(x)\}\,C\{g(x)\}$ $= F_C(\omega)G_C(\omega)$